Some facts about harmonic functions on $\mathbb{C}$ or $\mathbb{R}^2$

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1 holomorphic functions and harmonic functions

This note is about some basic fact of harmonic functions. From my point of view, some properties of holomorphic functions seems to be too mysterious. So here is some fact from PDE point of view to explain some of them from the aspect of real functions. There are few reasons which connects harmonic function with holomorphic functions, although holomorphic functions are more restrictive than harmonic functions.

Say f is holomorphic, then

1. $Re(f)$ and $Im(f)$ are harmonic.
2. $|f|^2$ is subharmonic.
3. $\log|f|$ is harmonic if $f \neq 0$ and is subharmonic in the weak sense in general.

In fact, there are much more stuff related to harmonic functions itself. If you are interested, you may take a look on the book "Harmonic function theory" by Sheldon Axler, Paul Bourdon and Wade Ramey.

Definition 1.1. *A function $u \in C^2(\Omega)$, $\Omega \subset \mathbb{C}$ is said to subharmonic (superharmonic) if $\Delta u \geq (\leq) 0$. In particular, u is harmonic if u is both subharmonic and superharmonic.*

Remark: We can still define harmonicity for a weaker class of functions, say $W^{2,p}$. You can google for more information.

1.1 Mean value property & Maximum principle on $\mathbb{R}^n$

Theorem 1.2. *(Mean value property)Let $u \in C^2(\Omega)$ with $\Delta u \geq 0$. Suppose $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$, then*

$$u(x_0) \leq \fint_{B(x_0,r)} u(x)\, dV \ , \ \fint_{\partial B(x_0,r)} u(x)\, dA.$$

Proof. Define

$$f(r) = \frac{1}{r^{n-1}} \int_{\partial B(x_0,r)} u(x)\, dA = \int_{\partial B(0,1)} u(a + r\omega)\, dA_1$$

We have

$$\begin{aligned} f'(r) = \int_{\partial B(0,1)} \frac{\partial u}{\partial n}|_{a+r\omega}\, dA_1 &= \frac{1}{r^{n-1}} \int_{\partial B(a,r)} \frac{\partial u}{\partial n}\, dA \\ &= \frac{1}{r^{n-1}} \int_{B(a,r)} \Delta u\, dV \geq 0. \end{aligned}$$

Hence, for all $r > \epsilon > 0$, $f(r) \geq f(\epsilon)$. Result follows from taking $\epsilon \to 0$ and the continuity of u. Another inequality follows from integrating over r. $\square$

Theorem 1.3. *(Strong Maximum Principle)Let $u \in C^2(\Omega)$ with $\Delta u \geq 0$. Suppose there exists $x_0 \in \Omega$ such that $u(x_0) = \sup_\Omega u(x)$. Then u is a constant function.*

Proof. From the mean value property of subharmonic function, if it achieves maximum somewhere, it is locally constant. Hence, the function is constant function by the connectedness of Ω. $\square$

The following global conclusion follows immediately.

Theorem 1.4. *(Weak Maximum principle) Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $\Delta u \geq 0$ and Ω being bounded in $\mathbb{R}^n$, then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Consequently, for harmonic function u,

$$\inf_{\partial\Omega} u \leq u(x) \leq \sup_{\partial\Omega} u, \quad \forall\, x \in \Omega.$$

Remark: If Ω is unbounded, the conclusion clearly fails. For example, take $u(x, y) = y$ on $\mathbb{R}^2$.

1.2 fundamental solution of Δ on $\mathbb{C}$

Recall the fundamental solution of $\Delta f = 0$ in dimension 2:

$$\Gamma(x - y) = \frac{1}{2\pi} \log |x - y|.$$

In particular, the fundamental solution goes to ∞ as $x \to \infty$. This property yields the following feature of dimension 2.

Theorem 1.5. *Let $u \in C^2(\mathbb{C})$ with $\Delta u \geq 0$. If $u = o(\log r)$, then u is constant.*

Proof. Consider $u_\epsilon = u - \epsilon \log r$ on $B(0, 1)^c$. Since $u = o(\log r)$, we have

$$u_\epsilon \to -\infty \text{ as } x \to \infty.$$

Therefore, u_ϵ must attain maximum somewhere. By weak maximum principle, u_ϵ achieves maximum on $\partial B(0, 1)$. That is to say:

$$u_\epsilon(x) \leq \max_{\partial B(0,1)} u, \quad \forall\, |x| \geq 1.$$

Taking $\epsilon \to 0$ and also use weak maximum principle again,

$$u(x) \leq \max_{\partial B(0,1)} u, \quad \forall\, x \in \mathbb{R}^2.$$

By strong maximum principle, u is constant.

Remark: In fact, the above property is still true on $\mathbb{C}^n$. $\square$

1.3 Green representation of harmonic function on $\mathbb{C}$

By using Green second identity, we infer that for harmonic function u,

$$u(y) = \frac{1}{2\pi} \int_{\partial\Omega} \left(u(x) \frac{\partial}{\partial n} \log|x-y| - \log|x-y| \frac{\partial u}{\partial n} \right) \, dA$$

The following conclusion follows immediately from the above expression.

Corollary 1.6. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $\Delta u = 0$. Then $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$.*

In fact, we have a more simple form if $\Omega = B(a, R)$. By translation and rescaling, we assume $a = 0$ and $R = 1$.

Theorem 1.7. *Suppose $f : \partial B(1) \to \mathbb{R}$ is a continuous function. Then there is continuous function $u : \bar{B}(1) \to \mathbb{R}$ such that*

1. *$u(z) = f(z)$ for $z \in \partial B(1)$.*
2. *u is harmonic in $B(1)$.*

Moreover, u is given by the following formula

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1-r^2}{1 - 2r\cos(\theta - t) + r^2} \right] u(e^{it}) \, dt$$

for $r \in [0, 1)$ and $0 \le \theta \le 2\pi$.

I just omit the proof here. A proof using holomorphic function will be inside your homework later. You may also find a proof using Fourier series in Stein's Fourier analysis. A proof using Green function can also be found in standard PDE textbook.

A simple consequence of this is the following regularity theorem:

Theorem 1.8. *If $u : \Omega \to \mathbb{R}$ is a continuous function with the mean value property. Then u is harmonic funciton. In particular, u is smooth function.*

Proof. Let $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$. And use above theorem to solve the dirichlet problem:

$$\Delta v = 0 \ \text{ and } \ v = u \text{ on } \partial B(x_0, r).$$

Since u satisfies Mean value property, also so is v. Therefore, for the function $f = v - u$. We can apply maximum principle to conclude that $\forall\, x \in B(x_0, r)$,

$$0 = \inf_{\partial B(x_0,r)} f \le f(x) \le \sup_{\partial B(x_0,r)} f = 0.$$

Hence, $u = v$ which is harmonic in the strong sense. $\square$